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LETTER TO THE EDITOR

Quantum algebras in classical mechanics

Sergio A Hojman†

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior, CU, 04510 México, DF, México

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Abstract. It is shown that the equations of motion do not determine their Hamiltonian descriptions uniquely. In particular, different Poisson bracket relations are admissible for a given physical problem. The free symmetric top and a charged particle moving on a homogeneous and constant magnetic field, for instance, may be described by a classical Hamiltonian theory with an $SU(2)_q$ Poisson bracket structure.

Quantum algebras have been much studied lately [1-6]. The purpose of this letter is to show that q-algebras are special cases of more general structures which arise when studying Hamiltonian descriptions of classical systems. In this letter, we derive some results which may also be obtained by using a general method (which will be explained in detail elsewhere [7]) and we use them to construct a few simple and physically relevant examples of systems for which the Hamiltonian theory may be cast in terms of Poisson brackets with a q-algebra structure.

The usual formulation of Hamiltonian dynamics is based on a Lagrangian description of the equations of motion of the (regular) system considered [8, 9]. Singular Lagrangian systems may also be dealt with using a procedure revised by Dirac [10].

In this letter, we construct Hamiltonian theories taking the equations of motion as a starting point without using a Lagrangian description (which in some cases it may even fail to exist). There has been much interest in this approach lately (mostly in the fields of fluid dynamics and plasma physics [11, 12]).

The Hamiltonian description is based on an antisymmetric matrix J^{ab} which represents the Poisson brackets between the (in general, non-canonical) coordinates in phase space and a Hamiltonian H. Given a system of first-order equations (if the original equations are not first-order they can always be recast in that way),

$$\dot{x}^a = f^a(x^b, t) \qquad a = 1, \ldots, N. \tag{1}$$

The Hamiltonian theory is given by J^{ab} and H such that

$$f^a = J^{ab}\frac{\partial H}{\partial x^b} \tag{2}$$

$$J^{ab} = -J^{ba} \tag{3}$$

$$J^{ab}{}_{,c}J^{cd} + J^{bd}{}_{,c}J^{ca} + J^{da}{}_{,c}J^{ab} = 0. \tag{4}$$

† Present address: Departamento de Física, Facultad de Ciencias Universidad de Chile, Casilla 653, Santiago, Chile.

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In other words, f^a may be written as

$$f^a = [x^a, H] \tag{5}$$

for the Poisson bracket structure defined by $[A, B] = (\partial A/\partial x^a)J^{ab}(\partial B/\partial x^b)$. Equation (4) is the Jacobi identity for such a bracket.

Equations (2) and (3) imply that

$$\frac{\partial H}{\partial x^a} f^a = 0 \tag{6}$$

which means that, if H does not depend explicitly on time, H is a constant of motion. Therefore, to construct a time-independent Hamiltonian, we need to find a constant of motion of (1). The construction of J^{ab} may, in general, be achieved using geometric arguments [7], but for the examples discussed below the general theory will not be needed.

We are interested in $SU(2)_q$ and we therefore consider the case $N = 3$. For any odd N we get

$$\det J = 0 \tag{7}$$

or

$$J^{ab} V_b = 0 \tag{8}$$

for some non-trivial V_b. It has been shown [13] that if J^{ab} satisfies (2) and (3), then V_b is always proportional to a gradient,

$$V_b = \lambda \frac{\partial C}{\partial x^b} \tag{9}$$

for some λ and C. It is straightforward to see that

$$[C, A] = 0 \tag{10}$$

for any $A(x^a)$. C is called a Casimir function for J^{ab} because it has vanishing Poisson bracket with any dynamical variable. If C is time independent, then C must be a constant of motion.

Conversely, it is direct to see that J^{ab} given by

$$J^{ab} = \mu \varepsilon^{abc} \frac{\partial C}{\partial x^c} \tag{11}$$

is antisymmetric and it satisfies the Jacobi identity for any function μ. Furthermore, C is a Casimir function for J^{ab}. It is worth noting that the fact that C is a Casimir function for J^{ab} is related to the construction of the Hamiltonian theory (and of J^{ab} in particular) and it is *not*, in any way, contained in (1). There is only a consistency requirement, i.e. that C be a constant of motion for (1), but *any* such constant will do.

Let us consider the free symmetric top. The equations of motion are

$$\dot{L}_1 = \frac{(I - I_3)}{II_3} L_2 L_3 \tag{12}$$

$$\dot{L}_2 = \frac{(I_3 - I)}{II_3} L_1 L_3 \tag{13}$$

$$\dot{L}_3 = 0 \tag{14}$$

where L_i are the components of angular momentum along principal axes and $I_1 = I_2 = I$ and I_3 are the eigenvalues of the inertia tensor of the top.

It is very well known that

$$K_1 = L_3 \tag{15}$$

and

$$K_2 = L_1^2 + L_2^2 \tag{16}$$

are constants of motion.

The standard Hamiltonian description is given by

$$J_1^{ab} = \varepsilon^{abc} L_c \tag{17}$$

$$H_1 = -\left(\frac{L_1^2 + L_2^2}{2I} + \frac{L_3^3}{2I_3}\right). \tag{18}$$

It is straightforward to check that (12)–(14) follow from (2) using the expressions (17) and (18). The bracket (17) satisfies (3) and (4).

It is worth noticing that J_1^{ab} and H_1 may be rewritten in terms of K_1 and K_2 as

$$J_1^{ab} = \tfrac{1}{2}\varepsilon^{abc} \frac{\partial(K_1^2 + K_2)}{\partial L_c} \tag{19}$$

and

$$H_1 = -\frac{1}{2}\left(\frac{K_2}{I} + \frac{K_1^2}{I_3}\right). \tag{20}$$

Therefore, C_1 given by

$$C_1 = K_1^2 + K_2 \tag{21}$$

is a Casimir operator for J_1^{ab} given by (17) (or (19)).

Now, for any Poisson bracket structure which admits Casimir functions, the Hamiltonian is not uniquely defined. In fact, an arbitrary function of the Casimir functions may be added to the original Hamiltonian without changing the equations of motion. In particular, we may consider

$$H_1' = H_1 + \frac{1}{2I} C_1 = \frac{(I_3 - I)}{2II_3} K_1^2 \tag{22}$$

which gives rise, of course, to the correct equations of motion (12)–(14) using the bracket (17).

Consider now a general Casimir function

$$C = C(K_1^2, K_2) \tag{23}$$

and define the bracket

$$J^{ab} = \mu\varepsilon^{abc} \frac{\partial C}{\partial L_c}. \tag{24}$$

It may be shown that J^{ab} satisfies conditions (3) and (4) for an arbitrary function μ. Note that the bracket defined by (24) need not (and will not, in general) define a Lie algebra. Consider an arbitrary function H

$$H = H(K_1^2, K_2) \tag{25}$$

such that

$$\Delta \equiv \frac{\partial C}{\partial K_2}\frac{\partial H}{\partial K_1^2} - \frac{\partial C}{\partial K_1^2}\frac{\partial H}{\partial K_2} \neq 0. \tag{26}$$

We now recover (12)-(14) by simply requiring that

$$\mu = \frac{(I - I_3)}{4II_3\Delta}. \tag{27}$$

We have therefore constructed a Hamiltonian theory for (12)-(14) which contains two arbitrary functions C and H subject to the condition (26).

To get a q-algebra $\mathrm{SU}(2)_q$ for the Poisson bracket relations consider the choice of the $\mathrm{SU}(2)_q$ classical Casimir function (for constant η)

$$C_2 = \left(L_1^2 + L_2^2 + \frac{\sinh \eta}{\eta} \left(\frac{\sinh(\eta L_3)}{\sinh \eta} \right)^2 \right) \tag{28}$$

and define H_2 by

$$H_2 = \frac{(I_3 - I)}{2II_3} L_3^2. \tag{29}$$

We then get, using (27)

$$\mu_2 = \tfrac{1}{2} \tag{30}$$

and the Poisson bracket relations

$$J_2^{ab} = \mu_2 \varepsilon^{abc} \frac{\partial C_2}{\partial L_c} \qquad J_2^{12} = \frac{1}{2}\frac{\sinh(2\eta L_3)}{\sinh \eta} \qquad J_2^{23} = L_1 \qquad J_2^{31} = L_2 \tag{31}$$

are exactly those of $\mathrm{SU}(2)_q$, in the classical (Poisson bracket) case.

We can now consider H_2' which differs from H_2 by an arbitrary function of C_2. The Hamiltonian H_2' generates (12)-(14) using the brackets J_2^{ab} given by (31). One example is

$$\begin{aligned} H_2' &= H_2 - \frac{1}{2I} C_2 \\ H_2' &= -\frac{L_1^2 + L_2^2}{2I} + \frac{(I_3 - I)}{2II_3} L_3^2 - \frac{\sinh \eta}{2I\eta} \left(\frac{\sinh(\eta L_3)}{\sinh \eta} \right)^2 \end{aligned} \tag{32}$$

which is in some sense analogous to H_1 in the SU(2) case. (H_2' reduces to H_1 in the limit $\eta \to 0$.)

H_2' given by (32) and the $\mathrm{SU}(2)_q$ brackets defined in (31) may be used to describe the symmetric top in a similar way in which H_1 and J_1^{ab} corresponding to SU(2) perform the same task.

It is a straightforward matter to convince oneself that the operator algebra induced by (31) as well as the operator Hamiltonian corresponding to (29) give rise to the correct quantum equations of motion for the symmetric top (using then the $\mathrm{SU}(2)_q$ algebra rather than SU(2) one for the commutators of angular momentum). Again, in the quantum mechanical case the Hamiltonian may be modified by the addition of an arbitrary function of the $\mathrm{SU}(2)_q$ Casimir operator and the quantum analogue of H_2' give rise to the correct quantum equations.

Similarly, the Hamiltonian description of the motion of a charged particle on a constant and homogeneous magnetic field $\boldsymbol{B}$ can be achieved with an $\mathrm{SU}(2)_q$ algebra. In fact, the equations of motion are

$$\dot{v}_1 = \alpha v_2 B \qquad (B = |\boldsymbol{B}|) \tag{33}$$

$$\dot{v}_2 = -\alpha v_1 B \tag{34}$$

$$\dot{v}_3 = 0 \tag{35}$$

where v_i are the components of the velocity, α is a constant and we have chosen the 3-axis in the $\boldsymbol{B}$ direction. We have that k_1 and k_2

$$k_1 = v_3 \qquad k_2 = v_1^2 + v_2^2$$

are constants of motion.

A particular choice of Hamiltonian is

$$L = -\alpha B v_3 \tag{36}$$

which works both for the $\mathrm{SU}(2)_q$ and SU(2) Poisson bracket relations for the components of the velocity. Other Hamiltonians are allowed by adding arbitrary functions of the Casimir functions of $\mathrm{SU}(2)_q$ (or SU(2)) depending on the Poisson bracket relations used in the Hamiltonian description. Again, the quantum version of the theory can be carried out with an $\mathrm{SU}(2)_q$ algebra for the commutation relations of the components of the velocity, using a procedure similar to the one described for the symmetric top.

We have thus showed that the Hamiltonian descriptions of the free symmetric top and of the charged particle moving on a constant and homogeneous magnetic field may be achieved using the $\mathrm{SU}(2)_q$ algebra for the Poisson bracket (or commutator) relations both in the classical and quantum mechanical cases. Note that more general algebras and Hamiltonians are allowed by (23)-(27) and (33)-(36).

It is worth mentioning that the use of q-algebras in nuclear energy spectra has improved previous theoretical results (see [5]). Strictly speaking, that model corresponds to a spherical rotator. The Hamiltonian used is proportional to the Casimir operator of $\mathrm{SU}(2)_q$ and, therefore, angular momentum is conserved.

It is interesting to consider applications of the ideas sketched in this letter to field theory as well as fluid dynamics. The construction of Hamiltonian descriptions for singular systems using the scheme presented here is also worth studying.

The general construction of the Hamiltonian description starting from the equations of motion as well as some applications to singular systems is done in [7]. The construction of a Hamiltonian theory for Bianchi V cosmological models is discussed in [14]. The applications of the ideas described in this letter to the construction of group algebras with given Casimir functions (or operators) is performed in [15].

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References

[1] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[2] De Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371

[3] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[4] Jimbo M 1986 *Commun. Math. Phys.* **102** 537
[5] Raychev P P, Roussev R P and Smirnov Yu F 1990 *J. Phys. G: Nucl. Part. Phys.* **16** L137
[6] Song X C 1990 *J. Phys. A: Math. Gen.* **23** L821
[7] Hojman S 1990 *Construction of Hamiltonian Formalisms for Differential Equations* in preparation
[8] Goldstein H 1980 *Classical Mechanics* (Menlo Park, CA: Addison-Wesley)
[9] Landau L and Lifschitz E M 1976 *Mechanics* (Oxford: Pergamon)
[10] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University)
[11] Morrison P 1982 *AIP Conf. Proc.* **88** 13
[12] Holm D D 1986 *Phys. Fluids* **29** 7
[13] Littlejohn R G 1982 *AIP Conf. Proc.* **28** 47
[14] Hojman S, Núñez D and Ryan M P Jr 1990 *Quantization of Bianchi V Cosmological Models* in preparation
[15] Castaños O and Hojman S 1990 *Do the Casimir Operators determine the Algebra?* in preparation